A conformally broken system as the constrained SL(2,R) WZNW model, its integrability and dressing symmetry

1994 J. Phys. A: Math. Gen. 27 6525

(http://iopscience.iop.org/0305-4470/27/19/022)

J. Phys. A: Math. Gen. 27 (1994) 6525–6532. Printed in the UK

A conformally broken system as the constrained $SL(2,\mathcal{R})$ WZNW model, its integrability and dressing symmetry

Bo-yu Hou†, Huan-xiong Yang‡, Wen-li Yang† and Yan-shen Wang†

† Institute of Modern Physics, Northwest University, Xi'an, 710069 People's Republic of China

‡ Zhejiang Institute of Modern Physics, Zhejiang University, Hangzhou, 310027 People's Republic of China

Received 22 April 1994

Abstract. This paper is devoted to the study of the conformally broken system related to the conformally invariant $SL(2,\mathcal{R})$ WZNW model. Our starting point is such an action that is similar in form to the action of gauged WZNW theories developed by Balog *et al*. However, the Lagrangian multipliers in our case are assumed to take their values on the mixing of upper and lower Borel subalgebras of $sl(2)$. The system obtained turns out to be an off-conformal extension of the sinh–Gordon model. Its integrability is displayed by Lax formalism with an arbitrary spectral parameter, the classical r-matrix, and the hidden dressing symmetry.

The two-dimensional WZNW models are amongst the most fundamental conformal field theories (CFTs), their gauged or constrained versions now attracting the attention of many researchers. In the framework of Hamiltonian reduction, some of the important conformally invariant systems, such as the Liouville model, Toda models, extended Toda models, conformally affine Toda models and the coset models of CFTs, have already been formulated as the gauged WZNW field systems [1–10]. Therefore these systems can be solved by means of resolving WZNW models. The other properties of these systems, for instance their conformal algebras (W-algebras) and correlation functions, could also be obtained with the help of WZNW models.

As is well known, the conformal field theory only describes the critical points of statistical systems. The scaling regions near criticality have to be described by some conformally broken theories whose ultraviolet fixed points are CFTs. In view of this fact, an important area of study has become the off-conformal behaviours for the systems described by CFTs. Particular interest centres on building in a non-perturbative way new theories from the conformal WZNW models that are no longer conformally invariant but that still keep their integrability [3, 8]. To our knowledge, an interesting advance which has been made in this direction can be characterized by the so-called Babelon–Bonora mechanism [5]. In terms of this mechanism, one should start with the two-loop WZNW models (based on affine Lie groups with a non-vanishing centre) and construct new CFTs via Hamiltonian reduction procedure, and then induce the off-conformal systems through the so-called 'spontaneous breaking' of the conformal symmetries. Relying on such an idea, the off-conformally integrable sinh–Gordon as well as the general affine Toda models have successfully been connected with the two-loop WZNW theories [6]. Moreover, the soliton solutions of these conformal broken systems have been obtained by virtue of the representations of the two-loop Kac–Moody algebras [9, 10].

0305-4470/94/196525+08$19.50

It goes without saying that the aforementioned Babelon–Bonora mechanism is very important in the development of non-perturbative off-conformally integrable field theories. However, it is still necessary to examine other methods for directly inducing the conformally broken integrable systems from conformal WZNW models. If such methods exist, we will enhance our understanding about the off-critical phenomena of WZNW models, and better describe the relevant phase transitions. Motivated by these facts, in the present paper we construct a constrained $SL(2,\mathcal{R})$ WZNW model. The classical action of the constrained system considered here has the same form as that of the gauged WZNW theories developed by Balog *et al* [4] except that in our case the Lagrangian multipliers $A_\pm(x)$ assume such forms that these $A_\pm(x)$ can no longer be explained as gauge fields. The constants μ, ν are also taken different forms from those in [4]. Hence what we are considering cannot be explained as a gauged $SL(2,\mathcal{R})$ WZNW theory. Just as we would expect, the physical ingredient of the constrained system is off-conformal invariance. This induced off-conformal system is neither the standard sinh–Gordon model nor the 'conformal $\widehat{sl}(2)$ Toda model' given by Babelon and Bonora, but an extension of the sinh–Gordon model. It is also shown that such an off-conformal system does continuously preserve its integrability, at least at the classical level.

Let us first briefly review the gauged $SL(2,\mathcal{R})$ WZNW theory of Balog *et al*. The starting point is the WZNW model based on the simple Lie group $SL(2,\mathcal{R})$ with a three-dimensional Lie algebra $sl(2)$

$$[H,E]=2E \qquad [H,F]=-2F \qquad [E,F]=H$$

$$\mathrm{Tr}(H^2)=2 \qquad \mathrm{Tr}(EF)=1$$

whose action is as follows:

$$S_{\mathrm{WZNW}}(g)=\frac{\kappa}{2}\int \mathrm{d}^2x\ \mathrm{Tr}(\partial_\mu g g^{-1}\partial^\mu g g^{-1})-\frac{\kappa}{3}\int_B \mathrm{d}^3x\,\epsilon^{ijk}\,\mathrm{Tr}(\partial_i g g^{-1}\partial_j g g^{-1}\partial_k g g^{-1}) \tag{1}$$

where the field $g(x)$ takes its value on $SL(2,,\mathcal{R})$ group manifold, and κ is a dimensional coupling constant. The famous Balog *et al* gauged WZNW theory is then defined by

$$I(g,A_\pm)=S_{\mathrm{WZNW}}(g)+\kappa\int \mathrm{d}^2x\ \mathrm{Tr}\left[A_-(\partial_+ g g^{-1}-\mu)+A_+(g^{-1}\partial_- g-\nu)+A_- g A_+ g^{-1}\right]$$

$$(\partial_\pm=\partial_0\pm\partial_1) \tag{2}$$

In (2) the constant μ, ν and the Lagrangian multipliers $A_\pm(x)$ are considered to take their values on the Borel subalgebras of $sl(2)$: $\mu=F$, $\nu=E$, $A_-=A_-^-E$ and $A_+=A_+^+F$. It is due to these choices of the Lagrangian multipliers that the action (2) is invariant under the local gauge transformations

$$g\to\alpha g\beta^{-1} \qquad A_-\to\alpha A_-\alpha^{-1}+\alpha\partial_-\alpha^{-1} \qquad A_+\to\beta A_+\beta^{-1}+\partial_+\beta\beta^{-1}$$

where

$$\alpha=\exp(\alpha(x)E) \qquad \beta=\exp(\beta(x)F).$$

It is obvious that the gauge invariant content is just defined on the subspace of the solution space of the $SL(2,\mathcal{R})$ WZNW model, which does naturally keep the original conformal symmetry.

Now we establish the conformally broken version of the above gauged WZNW theory. The expected off-conformal system is still defined by the action (2). But the constants μ, ν as well as Lagrangian multipliers $A_\pm(x)$ are in our case replaced by

$$\begin{cases}\mu=\sqrt{\gamma/2}\,(E+F)\\ A_-=A_-^+F+A_-^-E\end{cases}\qquad\begin{cases}\nu=\sqrt{\gamma/2}\,(E+F)\\ A_+=A_+^+F+A_+^-E\end{cases} \tag{3}$$

where γ is another non-vanishing constant. Owing to (3), the action (2) is no longer invariant under any gauge transformation. This is a striking difference between our system and that of Balog *et al.* The absence of local gauge symmetry also makes our system different from Park's off-conformal system described by integro-differential equations [3], which contain the sinh–Gordon equation as a special case. The equations of motion following from (2) and (3) read as

$$\begin{aligned}
&\partial_-(\partial_+ g g^{-1} + gA_+g^{-1}) + \partial_+ A_- - [\partial_+ g g^{-1} + gA_+g^{-1}, A_-] = 0\\
&\partial_+(g^{-1}\partial_- g + g^{-1}A_- g) + \partial_- A_+ - [A_+, g^{-1}\partial_- g + g^{-1}A_- g] = 0\\
&\mathrm{Tr}[E(\partial_+ g g^{-1} + gA_+g^{-1} - \mu)] = 0\\
&\mathrm{Tr}[F(\partial_+ g g^{-1} + gA_+g^{-1} - \mu)] = 0\\
&\mathrm{Tr}[E(g^{-1}\partial_- g + g^{-1}A_- g - \nu)] = 0\\
&\mathrm{Tr}[F(g^{-1}\partial_- g + g^{-1}A_- g - \nu)] = 0.
\end{aligned} \tag{4}$$

The last four of these equations are clearly constraints: the number of independent fields in the system considered is therefore three.

The WZNW model, which we start with, is based on the non-compact group manifold $SL(2, \mathcal{R})$. So the group-valued field $g(x)$ is allowed locally decomposed into [1]

$$g(x) = \exp(vE)\exp(\phi H)\exp(wF). \tag{5}$$

Such a factorizable form is usually called the 'Gaussian decomposition' of $g(x)$. With the group parameters v, ϕ and w appearing in (5), we discover that the explicit solutions of the above constraints are

$$\begin{aligned}
A_- &= -[\partial_- v - \sqrt{\gamma/2}\,\mathrm{e}^{2\phi} - v^2\mathcal{P}_v]E + \mathcal{P}_v F\\
A_+ &= \mathcal{P}_w E - [\partial_+ w - \sqrt{\gamma/2}\,\mathrm{e}^{2\phi} - w^2\mathcal{P}_w]F
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{P}_v &= \left[\sqrt{\gamma/2}\,(1+w^2) + 2w\partial_-\phi - \partial_- w\right]/(\exp(2\phi) + 2vw)\\
\mathcal{P}_w &= \left[\sqrt{\gamma/2}\,(1+v^2) + 2v\partial_+\phi - \partial_+ v\right]/(\exp(2\phi) + 2vw).
\end{aligned}$$

Then the equations of motion for our system can be recast as

$$\begin{aligned}
&\partial_+\partial_-\phi + (\gamma/2)\mathrm{e}^{2\phi} + \sqrt{\gamma/2}\,(v^2-1)\mathcal{P}_v - \partial_-(w\mathcal{P}_w) = 0\\
&\partial_+\mathcal{P}_v + 2(\partial_+\phi + \sqrt{\gamma/2}\,v - w\mathcal{P}_w)\mathcal{P}_v = 0\\
&\partial_-\mathcal{P}_w + 2(\partial_-\phi + \sqrt{\gamma/2}\,w - v\mathcal{P}_v)\mathcal{P}_w = 0
\end{aligned} \tag{6}$$

where the constraints have been eliminated. Note that in the limit $v = w = 0$, the first equation of (6) degenerates to the standard sinh–Gordon equation and the others hold as identities. The system described by (6) is therefore an extension of the sinh–Gordon model whose Lagrangian density is

$$\mathcal{L} = \kappa\left[\partial_+\phi\partial_-\phi - (\exp(2\phi) + 2vw)\mathcal{P}_v\mathcal{P}_w + \sqrt{\gamma/2}\,(\partial_- v + \partial_+ w) - (\gamma/2)\exp(2\phi)\right]. \tag{7}$$

The system (7) is expected to be off-conformal. This can directly be shown by examining the transform properties of (6) under conformal transformation. But we would rather assume another indirect way for our purpose. From (7) we see that the canonical momenta conjugate to the fields ϕ, v and w are

$$\pi_\phi = 2\kappa\,[\partial_0\phi - v\mathcal{P}_v - w\mathcal{P}_w] \qquad \pi_v = \kappa[\mathcal{P}_v + \sqrt{\gamma/2}\,] \qquad \pi_w = \kappa[\mathcal{P}_w + \sqrt{\gamma/2}\,] \tag{8}$$

respectively. The basic (equal-time) Poisson brackets of these canonical variables are as usual defined as

$$\{\phi(x), \pi_\phi(y)\} = \{v(x), \pi_v(y)\} = \{w(x), \pi_w(y)\} = \delta(x^1 - y^1).$$

Making use of these canonical variables and their Poisson brackets, we introduce the classical $SL(2, \mathcal{R})$ Kac–Moody currents in phase space

$$\begin{aligned}
j(H, x) &= \pi_\phi + 2v\pi_v + 2\kappa\partial_1\phi \\
j(E, x) &= \pi_v \\
j(F, x) &= -v\pi_\phi - v^2\pi_v + \pi_w \exp 2\phi + 2\kappa(\partial_1 v - v\partial_1\phi) \\
\widetilde{j}(H, x) &= -\pi_\phi - 2w\pi_w + 2\kappa\partial_1\phi \\
\widetilde{j}(E, x) &= w\pi_\phi + w^2\pi_w - \pi_v \exp 2\phi + 2\kappa(\partial_1 w - w\partial_1\phi) \\
\widetilde{j}(F, x) &= -\pi_w.
\end{aligned}$$

It is not difficult to see that these currents are subject to the following Poisson bracket algebra

$$\begin{aligned}
&\{j(A, x), j(B, y)\} = j([A, B], x)\delta(x^1 - y^1) + 2\kappa\,\mathrm{Tr}(AB)\delta'(x^1 - y^1) \\
&\{\widetilde{j}(A, x), \widetilde{j}(B, y)\} = \widetilde{j}([A, B], x)\delta(x^1 - y^1) - 2\kappa\,\mathrm{Tr}(AB)\delta'(x^1 - y^1) \\
&\{j(A, x), \widetilde{j}(B, y)\} = 0 \\
&\delta'(x^1 - y^1) = \partial_1\delta(x^1 - y^1)
\end{aligned} \tag{9}$$

where A and B are two arbitrary elements of the $sl(2)$ algebra. In terms of the basic fields ϕ, v and w, the Kac–Moody currents can also be recast as

$$\begin{aligned}
&\begin{cases}
j(H, x) = 2\kappa[\partial_+\phi + \sqrt{\gamma/2}\,v - w\mathcal{P}_w] \\
j(E, x) = \kappa[\mathcal{P}_v + \sqrt{\gamma/2}\,] \\
j(F, x) = \kappa[-\partial_- v + v^2\mathcal{P}_v + \sqrt{\gamma/2}\,(\mathrm{e}^{2\phi} + 1)]
\end{cases} \\
&\begin{cases}
\widetilde{j}(H, x) = -2\kappa[\partial_-\phi + \sqrt{\gamma/2}\,w - v\mathcal{P}_v] \\
\widetilde{j}(E, x) = -\kappa[-\partial_+ w + w^2\mathcal{P}_w + \sqrt{\gamma/2}\,(\mathrm{e}^{2\phi} + 1)] \\
\widetilde{j}(F, x) = -\kappa[\mathcal{P}_w + \sqrt{\gamma/2}\,].
\end{cases}
\end{aligned} \tag{10}$$

Thus the equations of motion for the system (7) could yet be formulated into either

$$\begin{aligned}
&\partial_- J_+ - \partial_+ J_- + [J_+, J_-] = 0 \\
&J_+(x) = (1/2\kappa)j(H, x)H + \sqrt{\gamma/2}\,(E + F) \\
&J_-(x) = [(1/\kappa)j(F, x) - \sqrt{\gamma/2}\,]E + [(1/\kappa)j(E, x) - \sqrt{\gamma/2}\,]F
\end{aligned}$$

or

$$\partial_+ \widetilde{J}_- - \partial_- \widetilde{J}_+ + [\widetilde{J}_-, \widetilde{J}_+] = 0$$
$$\widetilde{J}_-(x) = (1/2\kappa)\widetilde{j}(H, x)H - \sqrt{\gamma/2}(E + F)$$
$$\widetilde{J}_+(x) = -[(1/\kappa)\widetilde{j}(F, x) + \sqrt{\gamma/2}]E - [(1/\kappa)\widetilde{j}(E, x) - \sqrt{\gamma/2}]F.$$

These two zero-curvature equations make it clear that the Kac–Moody currents in the considered system do not obey the chiral conservation laws. Therefore, such a system must have lost the conformal symmetry, although it came from the conformally invariant $SL(2, \mathcal{R})$ WZNW model.

We are now obliged to investigate the integrability for our system (7). This is because we are only interested in such off-conformal systems that continue to be completely integrable [11]. This issue will be discussed from the inverse scattering points. Enlightened by the algebraic structures of above zero-curvature equations, we find out that the equations of motion (6) have the following Lax representation:

$$\left[\partial_0 - \widetilde{M}(x, \lambda), \partial_1 - \widetilde{L}(x, \lambda)\right] = 0 \tag{11}$$

$$\begin{aligned}\widetilde{M}(x, \lambda) &= (1/4\kappa)j(H, x)H + \tfrac{1}{2}\left[((1+\lambda)/\lambda)\sqrt{\gamma/2} - (1/\kappa)j(F, x)\right]E \\ &\quad + \tfrac{1}{2}\left[((1+\lambda)/\lambda)\sqrt{\gamma/2} - (1/\kappa)j(E, x)\right]\lambda F \\ \widetilde{L}(x, \lambda) &= (1/4\kappa)j(H, x)H + \tfrac{1}{2}\left[((1-\lambda)/\lambda)\sqrt{\gamma/2} + (1/\kappa)j(F, x)\right]E \\ &\quad + \tfrac{1}{2}\left[((1-\lambda)/\lambda)\sqrt{\gamma/2} + (1/\kappa)j(E, x)\right]\lambda F\end{aligned} \tag{12}$$

with which an arbitrary spectral parameter λ has been decorated non-trivially. The presence of the parameter λ in the Lax pair is extremely important for showing integrability, which would imply that the symmetry hidden in the considered system is described by an infinite-dimensional loop algebra.

It then follows, from the classical Kac–Moody current algebra (9), that:

$$\begin{aligned}\left\{\widetilde{L}(x, \lambda) \overset{\otimes}{,} \widetilde{L}(y, \mu)\right\} &= \left[\widetilde{r}(\lambda, \mu), \widetilde{L}(x, \lambda) \otimes 1 + 1 \otimes \widetilde{L}(y, \mu)\right]\delta(x^1 - y^1) \\ &\quad - \left[\widetilde{s}(\lambda, \mu), \widetilde{L}(x, \lambda) \otimes 1 - 1 \otimes \widetilde{L}(y, \mu)\right]\delta(x^1 - y^1) - 2s(\lambda, \mu)\delta'(x^1 - y^1)\end{aligned} \tag{13}$$

where

$$\begin{aligned}\widetilde{r}(\lambda, \mu) &= \frac{1}{4\kappa}\left[\frac{1}{2}\frac{\lambda + \mu - 2\lambda\mu}{\lambda - \mu}H \otimes H + \frac{\mu(2 - \lambda - \mu)}{\lambda - \mu}E \otimes F + \frac{\lambda(2 - \lambda - \mu)}{\lambda - \mu}F \otimes E\right] \\ \widetilde{s}(\lambda, \mu) &= -\frac{1}{4\kappa}[\tfrac{1}{2}H \otimes H + \mu E \otimes F + \lambda F \otimes E].\end{aligned} \tag{14}$$

After lengthy but straightforward computation we see that these $\widetilde{r}(\lambda, \mu)$ and $\widetilde{s}(\lambda, \mu)$ satisfy the so-called extended Yang–Baxter equation [12]

$$\begin{aligned}&\left[(\widetilde{r} + \widetilde{s})_{23}(\nu, \eta), (\widetilde{r} + \widetilde{s})_{12}(\lambda, \mu)\right] + \left[(\widetilde{r} + \widetilde{s})_{23}(\nu, \eta), (\widetilde{r} + \widetilde{s})_{13}(\lambda, \eta)\right] \\ &\quad + \left[(\widetilde{r} + \widetilde{s})_{13}(\lambda, \eta), (\widetilde{r} - \widetilde{s})_{12}(\lambda, \mu)\right] = 0.\end{aligned}$$

So the fundamental Poisson bracket (13) satisfies the Jacobian identity, and the system (7) is thereby a completely integrable system [12, 13].

Despite the fact that the bracket (13) assumes a non-ultralocal form, the system under consideration *is virtually an ultralocally integrable system.* In terms of the canonical expressions of Kac–Moody currents, listed before equation (9), one can easily determine that in phase space the Lax operator $\widetilde{L}(x,\lambda)$ in (12) can be divided into two parts $\widetilde{L}=\widetilde{L}^{(0)}+\widetilde{L}^{(1)}$ of which $\widetilde{L}^{(0)}$ *is defined only as the sum of terms that are not involved in the spatial* derivatives of the canonical variables. Correspondingly

$$\widetilde{L}^{(1)}=\tfrac{1}{2}\partial_1\phi H+(\partial_1 v-v\partial_1\phi)E.$$

The space-derivatives of the fields ϕ and v in $\widetilde{L}^{(1)}$ result in the $\delta'(x^1-y^1)$-term appearing in the fundamental Poisson bracket (13), which is the very reason why (13) is non-ultralocal. Fortunately, this non-ultralocality for the system (7) is merely an outward non-ultralocality. Note that $\widetilde{L}^{(1)}$ exhibits as the spatial component of a pure gauge potential $\widetilde{L}^{(1)}=\partial_1\alpha\alpha^{-1}$ $[\alpha\equiv\exp(vE)\exp(\frac{\phi}{2}H)]$. Hence the above non-ultralocality can be removed via the following symmetric gauge transformation for zero-curvature equation (11):

$$\widetilde{M}\longrightarrow M=\alpha^{-1}\widetilde{M}\alpha+\partial_0\alpha^{-1}\alpha \qquad \widetilde{L}\longrightarrow L=\alpha^{-1}\widetilde{L}\alpha+\partial_1\alpha^{-1}\alpha.$$

Consequently we get an alternative Lax representation for the system (7), namely

$$[\partial_0-M(x,\lambda),\partial_1-L(x,\lambda)]=0 \tag{15}$$

$$\begin{aligned}M(x,\lambda)=\tfrac{1}{2}\Big[&(\partial_1\phi-w\mathcal{P}_w)H+e^{\phi}(\mathcal{P}_w-\sqrt{\gamma/2})E-e^{-\phi}(v^2\mathcal{P}_v+\sqrt{\gamma/2})E\\&+\sqrt{\gamma/2}e^{-\phi}(1/\lambda E)+\sqrt{\gamma/2})e^{\phi}F\Big]+\tfrac{1}{2}\lambda\mathcal{P}_v(vH+v^2e^{-\phi}E-e^{\phi}F)\\L(x,\lambda)=\tfrac{1}{2}\Big[&(\partial_0\phi-w\mathcal{P}_w)H+e^{\phi}(\mathcal{P}_w+\sqrt{\gamma/2})E+e^{-\phi}(v^2\mathcal{P}_v-\sqrt{\gamma/2})E\\&+\sqrt{\gamma/2}e^{-\phi}(1/\lambda E)+\sqrt{\gamma/2})e^{\phi}F\Big]-\tfrac{1}{2}\lambda\mathcal{P}_v(vH+v^2e^{-\phi}E-e^{\phi}F).\end{aligned} \tag{16}$$

As we expect, this Lax representation does yield an ultralocal fundamental Poisson bracket

$$\{L(x,\lambda)\overset{\otimes}{,}L(y,\mu)\}=[r(\lambda,\mu),L(x,\lambda)\otimes 1+1\otimes L(y,\mu)]\delta(x^1-y^1) \tag{17}$$

where the matrix $r(\lambda,\mu)$ is found to be

$$r(\lambda,\mu)=\frac{1}{2\kappa}\left[\frac{1}{4}\frac{\lambda+\mu-2\lambda\mu}{\lambda-\mu}H\otimes H+\frac{\mu(1-\lambda)}{\lambda-\mu}E\otimes F+\frac{\lambda(1-\mu)}{\lambda-\mu}F\otimes E\right] \tag{18}$$

which is undoubtedly a solution of the classical Yang–Baxter equation

$$[r_{12}(\lambda,\mu),r_{13}(\lambda,\eta)]+[r_{12}(\lambda,\mu),r_{23}(\mu,\eta)]+[r_{13}(\lambda,\eta),r_{23}(\mu,\eta)]=0. \tag{19}$$

By setting $(\lambda/(1-\lambda))\rightarrow\lambda$, $(\mu/(1-\mu))\rightarrow\mu$, we see that the r-matrix (18) is just a trigonometric solution of (19) [14].

The fundamental Poisson bracket (17) has an equivalent but more instructive description, if we introduce the transport matrix $T(x,\lambda)=\mathcal{P}\exp[\int_0^x dx\, L(x,\lambda)]$

$$\{T(x,\lambda)\overset{\otimes}{,}T(x,\mu)\}=[r(\lambda,\mu),T(x,\lambda)\otimes T(x,\mu)]. \tag{20}$$

It is then obvious that $\mathrm{Tr}(T(x,\lambda))$ generates an infinite number of quantities in involution. This conclusion achieves the full proof of the classical integrability of the system (7).

On the other hand, the integrability of the system (7) can alternatively be demonstrated by showing the nonlinear equations (6) have a (hidden) dressing symmetry. This means looking for the so-called dressing transformations for our system which are such gauge transformations that preserve the form of the Lax connection (16). The key step to this end is to define the relevant Riemann–Hilbert (factorization) problem. Since the Lax connection (16) is defined on the loop algebra $\widetilde{sl}(2)$

$$X(\lambda) = \sum_{i=-\infty}^{+\infty} \lambda^i (a_i H + b_i E + c_i F)$$

the subalgebras $g_\pm$ of this loop algebra entering the Riemann–Hilbert problem are specified in a manner similar to that used for the Heisenberg model [15]:

$$X_\pm(\lambda) = R_\pm(X(\lambda)) = \pm 2\kappa \oint_{C_\pm} \frac{\mathrm{d}\mu}{2\pi \mathrm{i}} (1/\mu(1-\mu)) \operatorname{Tr}_2[r^\pm(\lambda,\mu) 1 \otimes X(\mu)] \tag{21}$$

where the integration contour C_- encircles two singularities $\mu = 0, 1$, but C_+ only encircles one singularity $\mu = \infty$. $r^\pm$ are projection operators, which respectively correspond to the expansions of $r(\lambda, \mu)$ defined in (18), either in powers of (λ/μ) or in powers of (μ/λ). With respect to this explanation, we have

$$X_+(\lambda) = \sum_{i=0}^{+\infty} \lambda^i (a_i H + b_i E + c_i F) - \left[\left(a_0 + \frac{1}{2}\sum_{i=1}^{+\infty} a_i\right) H + \left(\sum_{i=0}^{\infty} b_i\right) E + c_0 F\right]$$

$$X_-(\lambda) = -\sum_{i=-\infty}^{-1} \lambda^i (a_i H + b_i E + c_i F) - \left[\left(a_0 + \frac{1}{2}\sum_{i=1}^{+\infty} a_i\right) H + \left(\sum_{i=0}^{\infty} b_i\right) E + c_0 F\right].$$

It is easy to check that $g_\pm = \operatorname{Im} R_\pm$ indeed constitute two subalgebras of $\widetilde{sl}(2)$. Besides any $X(\lambda) \in \widetilde{sl}(2)$ has a unique decomposition,

$$X(\lambda) = X_+(\lambda) - X_-(\lambda).$$

This is the definition of our Riemann–Hilbert problem.

In turn we look for the dressing transformations for Lax connection (16). Let $X(\lambda)$ be an element of $\widetilde{sl}(2)$ and set

$$\Theta_X(x, \lambda) \equiv (TXT^{-1})(x, \lambda) = \Theta_+(x, \lambda) - \Theta_-(x, \lambda). \tag{22}$$

Then $\Theta_X(x, \lambda)$ is also an element of $\widetilde{sl}(2)$

$$\Theta_X(x, \lambda) = \sum_{i=-\infty}^{+\infty} \lambda^i (x_i H + y_i E + z_i F). \tag{23}$$

In (22) $T(x, \lambda)$ is the aforesaid transport matrix which obeys the following linearly differential equations

$$\partial_0 T(x, \lambda) = M(x, \lambda) T(x, \lambda) \qquad \partial_1 T(x, \lambda) = L(x, \lambda) T(x, \lambda). \tag{24}$$

A dressing transformation is a gauge transformation for Lax connection with either Θ_+ or Θ_-, whose infinitesimal form is

$$\delta_X M = \partial_0 \Theta_+ - [M, \Theta_+] \qquad \delta_X L = \partial_1 \Theta_+ - [L, \Theta_+]. \tag{25}$$

The gauge transformation with Θ_- gives the same results. $X(\lambda)$ is usually taken to be spacetime independent. Thereby, it follows from (22) and (24) that

$$\partial_0 \Theta_X = [M, \Theta_X] \qquad \partial_1 \Theta_X = [L, \Theta_X]. \tag{26}$$

Inserting (16), (22) into (25) and (26) we get

$$\delta_X \phi = \sum_{i=1}^{+\infty} x_i \qquad \delta_X v = -2v x_0 - v^2 \mathrm{e}^{-\phi} z_0 + \mathrm{e}^{\phi} \sum_{i=0}^{+\infty} y_i \qquad \delta_X w = \mathrm{e}^{\phi} \sum_{i=1}^{+\infty} z_i. \tag{27}$$

These are the infinitesimal dressing transformations for the basic fields ϕ, v and w, in which the transform parameters x_i, y_i and z_i $(i = 0, \pm 1, \pm 2, \ldots)$ are some non-local functions of the basic fields.

Straightforward computations show that (27) are indeed symmetric transformations for equations of motion of our system (7). Moreover, they do preserve the form of Lax connection (16). Hence there is an infinite-dimensional symmetry hidden in our system, which exposes the integrability of (7) once again.

It is worth emphasizing that the generator of the dressing transformations is the transport matrix $T(x, \lambda)$. This conclusion is valid when we note that

$$
\begin{aligned}
&\{L(x,\lambda), \phi(y)\} = -(1/4\kappa) H \delta(x^1 - y^1) \\
&\{L(x,\lambda), v(y)\} = -(1/2\kappa)[(1-\lambda)v(x)H + (1-\lambda)v^2(x)\mathrm{e}^{-\phi(x)}E + \lambda \mathrm{e}^{\phi(x)}F]\delta(x^1 - y^1) \\
&\{L(x,\lambda), w(y)\} = -(1/2\kappa)\mathrm{e}^{\phi(x)} E \delta(x^1 - y^1).
\end{aligned}
\tag{28}
$$

By combining these brackets with (24), (27), the dressing transformations for the basic fields are rewritten into

$$
\begin{aligned}
&\delta_X \phi(y) = 2\kappa \oint_{C_-} \frac{\mathrm{d}\lambda}{2\pi \mathrm{i}} (1/\lambda(1-\lambda)) \operatorname{Tr}\left[X(\lambda) T^{-1}(x,\lambda)\{T(x,\lambda), \phi(y)\}\right] \\
&\delta_X v(y) = 2\kappa \oint_{C_-} \frac{\mathrm{d}\lambda}{2\pi \mathrm{i}} (1/\lambda(1-\lambda)) \operatorname{Tr}\left[X(\lambda) T^{-1}(x,\lambda)\{T(x,\lambda), v(y)\}\right] \\
&\delta_X w(y) = 2\kappa \oint_{C_-} \frac{\mathrm{d}\lambda}{2\pi \mathrm{i}} (1/\lambda(1-\lambda)) \operatorname{Tr}\left[X(\lambda) T^{-1}(x,\lambda)\{T(x,\lambda), w(y)\}\right] \\
&\qquad\qquad (x > y).
\end{aligned}
\tag{29}
$$

These formulae explicitly display the generation relationship of the dressing $\delta_X \phi$, $\delta_X v$ and $\delta_X w$ by $T(x, \lambda)$ (in a nonlinear way).

Acknowledgments

The authors are grateful to P Wang, K J Shi and L Chao for fruitful discussions. H X Yang would also like to acknowledge discussions with Y X Chen and K Lie.

References

[1] Alekseev A and Shatashvili S 1989 *Nucl. Phys.* B **323** 719
[2] Bershadsky M and Ooguri H 1989 *Commun. Math. Phys.* **126** 49
[3] Park Q-H 1990 *Nucl. Phys.* B **333** 267
[4] Balog J, Feher L, O'Raifeartaigh L, Forgacs P and Wipf A 1990 *Ann. Phys.* **203** 76; 1989 *Phys. Lett.* **227B** 214
[5] Babelon O and Bonora L 1991 *Phys. Lett.* **244B** 220
[6] Aratyn H, Ferreira L A, Gomes J F and Zimerman A H 1991 *Phys. Lett.* **254B** 372
[7] Hou B Y and Chao L 1992 *Int. J. Mod. Phys.* A **7** 7015
[8] Bonora L, Matrellini M and Zhang Y Z 1991 *Phys. Lett.* **253B** 373
[9] Ferreira L A, Gomes J F, Schwimmer A and Zimerman A H 1992 *Phys. Lett.* B **274** 65
[10] Olive D I, Turok N and Underwood J W R 1993 *Nucl. Phys.* B **401** 663
[11] Zamolodchikov A B 1986 *Sov. J. Nucl. Phys.* **44** 529; 1987 *JETP Lett.* **46** 160; 1988 *Int. J. Mod. Phys.* A **3** 743; 1989 *Int. J. Mod. Phys.* A **4** 4235
[12] Maillet J 1986 *Nucl. Phys.* B **269** 54
[13] Faddeev L D and Takhtajan L A 1986 *Hamiltonian Methods in the Theory of Solitons* (Berlin: Springer)
[14] Belavin A A and Drinfel'd V G 1984 *Sov. Rev.* C **4** 93
[15] Babelon O and Bernard D 1992 *Commun. Math. Phys.* **149** 279